Shape invariant hypergeometric type operators with application to quantum mechanics

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Received ; revised

Abstract: A hypergeometric type equation satisfying certain conditions defines either a finite or an infinite system of orthogonal polynomials. The associated special functions are eigenfunctions of some shape invariant operators. These operators can be analysed together and the mathematical formalism we use can be extended in order to define other shape invariant operators. All the considered shape invariant operators are directly related to Schrodinger type equations.

Keywords: orthogonal polynomials, associated special functions, shape invariant operators, raising and lowering operators, Schrödinger equation
PACS (2000): 02.30.Gp, 03.65.-w

1 Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [7], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \tag{2}$$

by choosing a function ϱ such that $(\sigma\varrho)' = \tau\varrho$.

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The equation (1) is usually considered on an interval (a,b), chosen such that

$$\begin{array}{rcl} \sigma(s)>0 & \text{for all} & s\in(a,b) \\ \varrho(s)>0 & \text{for all} & s\in(a,b) \\ \lim_{s\rightarrow a}\sigma(s)\varrho(s) & = & \lim_{s\rightarrow b}\sigma(s)\varrho(s)=0. \end{array} \tag{3}$$

Since the form of the equation (1) is invariant under a change of variable $s\mapsto cs+d$, it is sufficient to analyse the cases presented in table 1. Some restrictions are to be imposed to α, β in order the interval (a,b) to exist. The equation (1) defines an infinite sequence of orthogonal polynomials in the case $\sigma(s)\in\{1,\ s,\ 1-s^2\}$, and a finite one in the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$.

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	α,β	(a,b)
1	$\alpha s+\beta$	$\mathrm{e}^{\alpha s^2/2+\beta s}$	$\alpha<0$	$\mathbb{R}$
s	$\alpha s+\beta$	$s^{\beta-1}\mathrm{e}^{\alpha s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
$1-s^2$	$\alpha s+\beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$\alpha<\beta<-\alpha$	$(-1,1)$
s^2-1	$\alpha s+\beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$-\beta<\alpha<0$	$(1,\infty)$
s^2	$\alpha s+\beta$	$s^{\alpha-2}\mathrm{e}^{-\beta/s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
s^2+1	$\alpha s+\beta$	$(1+s^2)^{\alpha/2-1}\mathrm{e}^{\beta\arctan s}$	$\alpha<0$	$\mathbb{R}$

Table 1 The main cases

The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. A unified approach is not possible without a unified definition for the associated special functions. In this paper we define them as

$$\Phi_{l,m}(s)=\left(\sqrt{\sigma(s)}\right)^m\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s) \tag{4}$$

where Φ_l are the orthogonal polynomials defined by equation (1). The table 1 allows one to pass in each case from our parameters α, β to the parameters used in different approach.

In [2, 3] we presented a systematic study of the Schrödinger equations exactly solvable in terms of associated special functions. In the present paper, based on the factorization method [1, 5] and certain results of Jafarizadeh and Fakhri [6], we extend our unified formalism by adding other shape invariant operators.

2 Orthogonal polynomials

Let $\tau(s)=\alpha s+\beta$ be a fixed polynomial, and let

$$\lambda_l = -\frac{\sigma''(s)}{2}l(l-1)-\tau'(s)l = -\frac{\sigma''}{2}l(l-1)-\alpha\, l \tag{5}$$

for any $l\in\mathbb{N}$. It is well-known [7] that for $\lambda=\lambda_l$, the equation (1) admits a polynomial solution $\Phi_l=\Phi_l^{(\alpha,\beta)}$ of at most l degree

$$\sigma(s)\Phi_l''+\tau(s)\Phi_l'+\lambda_l\Phi_l=0. \tag{6}$$

If the degree of the polynomial Φ_l is l then it satisfies the Rodrigues formula [7]

$$\Phi_l(s)=\frac{B_l}{\varrho(s)}\frac{\mathrm{d}^l}{\mathrm{d}s^l}[\sigma^l(s)\varrho(s)] \tag{7}$$

where B_l is a constant. Based on the relation

$$\begin{array}{l} \{\ \delta\in\mathbb{R}\ |\ \lim_{s\to a}\sigma(s)\varrho(s)s^\delta=\lim_{s\to b}\sigma(s)\varrho(s)s^\delta=0\ \}\\ \qquad = \begin{cases} [0,\infty) & \text{if } \sigma(s)\in\{1,\ s,\ 1-s^2\}\\ [0,-\alpha) & \text{if } \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{cases}\end{array} \tag{8}$$

one can prove [3, 7] that the system of polynomials $\{\Phi_l\ |\ l<\Lambda\}$, where

$$\Lambda=\begin{cases} \infty & \text{for } \sigma(s)\in\{1,\ s,\ 1-s^2\}\\ \frac{1-\alpha}{2} & \text{for } \sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}\end{cases} \tag{9}$$

is orthogonal with weight function $\varrho(s)$ in (a,b). This means that equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0,\quad \Phi_1,\quad \Phi_2,\ \ldots$$

in the case $\sigma(s)\in\{1,\ s,\ 1-s^2\}$, and a finite one

$$\Phi_0,\quad \Phi_1,\quad \ldots,\quad \Phi_L$$

with $L=\max\{l\in\mathbb{N}\ |\ l<(1-\alpha)/2\}$ in the case $\sigma(s)\in\{s^2-1,\ s^2,\ s^2+1\}$.

The polynomials $\Phi_l^{(\alpha,\beta)}$ can be expressed (up to a multiplicative constant) in terms of the classical orthogonal polynomials as

$$\Phi_l^{(\alpha,\beta)}(s)=\begin{cases}\mathbf{H}_l\left(\sqrt{\frac{-\alpha}{2}}\,s-\frac{\beta}{\sqrt{-2\alpha}}\right) & \text{in the case}\quad \sigma(s)=1\\ \mathbf{L}_l^{\beta-1}(-\alpha s) & \text{in the case}\quad \sigma(s)=s\\ \mathbf{P}_l^{(-(\alpha+\beta)/2-1,\ (-\alpha+\beta)/2-1)}(s) & \text{in the case}\quad \sigma(s)=1-s^2\\ \mathbf{P}_l^{((\alpha-\beta)/2-1,\ (\alpha+\beta)/2-1)}(-s) & \text{in the case}\quad \sigma(s)=s^2-1\\ \left(\frac{s}{\beta}\right)^l\mathbf{L}_l^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{in the case}\quad \sigma(s)=s^2\\ \mathrm{i}^l\mathbf{P}_l^{((\alpha+\mathrm{i}\beta)/2-1,\ (\alpha-\mathrm{i}\beta)/2-1)}(\mathrm{i}s) & \text{in the case}\quad \sigma(s)=s^2+1\end{cases}\tag{10}$$

where $\mathbf{H}_l$, $\mathbf{L}_l^p$ and $\mathbf{P}_l^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively. The relation (10) does not have a very simple form. In certain cases we have to consider the classical polynomials outside the interval where they are orthogonal or for complex values of parameters.

3 Associated special functions. Shape invariant operators

Let $l\in\mathbb{N}$, $l<\Lambda$, and let $m\in\{0,1,...,l\}$. The functions

$$\Phi_{l,m}(s)=\kappa^m(s)\frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s)\qquad\text{where}\qquad\kappa(s)=\sqrt{\sigma(s)}\tag{11}$$

are called the *associated special functions.* If we differentiate (6) m times and then multiply the obtained relation by $\kappa^m(s)$ then we get the equation

$$H_m\Phi_{l,m}=\lambda_l\Phi_{l,m}\tag{12}$$

where H_m is the differential operator

$$\begin{aligned}H_m=&-\sigma(s)\tfrac{d^2}{ds^2}-\tau(s)\tfrac{d}{ds}+\tfrac{m(m-2)}{4}\tfrac{(\sigma'(s))^2}{\sigma(s)}\\ &+\tfrac{m\tau(s)}{2}\tfrac{\sigma'(s)}{\sigma(s)}-\tfrac{1}{2}m(m-2)\sigma''(s)-m\tau'(s).\end{aligned}\tag{13}$$

For each $m<\Lambda$, the special functions $\Phi_{l,m}$ with $m\leq l<\Lambda$ are orthogonal with respect to the scalar product

$$\langle f,g\rangle=\int_a^b\overline{f(s)}\,g(s)\varrho(s)ds\tag{14}$$

and the functions corresponding to consecutive values of m are related through the raising/lowering operators [2, 3]

$$\begin{aligned}A_m&=\kappa(s)\tfrac{d}{ds}-m\kappa'(s)\\ A_m^+&=-\kappa(s)\tfrac{d}{ds}-\tfrac{\tau(s)}{\kappa(s)}-(m-1)\kappa'(s)\end{aligned}\tag{15}$$

namely,

$$A_m\Phi_{l,m} = \begin{cases} 0 & \text{for } l=m \\ \Phi_{l,m+1} & \text{for } m<l<\Lambda \end{cases} \tag{16}$$
$$A_m^+\Phi_{l,m+1} = (\lambda_l-\lambda_m)\Phi_{l,m} \quad \text{for} \quad 0\le m<l<\Lambda.$$

Up to a multiplicative constant

$$\Phi_{l,m}(s) = \begin{cases} \kappa^l(s) & \text{for} \quad m=l \\ \frac{A_m^+}{\lambda_l-\lambda_m}\frac{A_{m+1}^+}{\lambda_l-\lambda_{m+1}}\cdots\frac{A_{l-1}^+}{\lambda_l-\lambda_{l-1}}\kappa^l(s) & \text{for} \quad m<l \end{cases} \tag{17}$$

and the operators H_m are shape invariant [2, 3]

$$\begin{array}{ll} H_m-\lambda_m = A_m^+A_m & A_mH_m = H_{m+1}A_m \\ H_{m+1}-\lambda_m = A_mA_m^+ & H_mA_m^+ = A_m^+H_{m+1}. \end{array} \tag{18}$$

The functions

$$\phi_{l,m} = \Phi_{l,m}/||\Phi_{l,m}|| \tag{19}$$

where $||f|| = \sqrt{\langle f,f\rangle}$ are the *normalized associated special functions*. Since [2, 3]

$$||\Phi_{l,m+1}|| = \sqrt{\lambda_l-\lambda_m}\,||\Phi_{l,m}|| \tag{20}$$

they satisfy the relations

$$\begin{array}{l} A_m\ \phi_{l,m} = \begin{cases} 0 & \text{for } l=m \\ \sqrt{\lambda_l-\lambda_m}\ \phi_{l,m+1} & \text{for } m<l<\Lambda \end{cases} \\ A_m^+\ \phi_{l,m+1} = \sqrt{\lambda_l-\lambda_m}\ \phi_{l,m} \quad \text{for } 0\le m<l<\Lambda \end{array} \tag{21}$$

$$\phi_{l,m} = \frac{A_m^+}{\sqrt{\lambda_l-\lambda_m}}\frac{A_{m+1}^+}{\sqrt{\lambda_l-\lambda_{m+1}}}\cdots\frac{A_{l-1}^+}{\sqrt{\lambda_l-\lambda_{l-1}}}\phi_{l,l}.$$

4 Application to Schrödinger type equations

It is well-known [5] that the equations $H_m\Phi_{l,m} = \lambda_l\Phi_{l,m}$ are directly related to certain Schródinger type equations. If in equation satisfied by $\Phi_{l,m}$

$$\begin{array}{c} -\sigma(s)\frac{d^2}{ds^2}\Phi_{l,m}(s) - \tau(s)\frac{d}{ds}\Phi_{l,m}(s) + [\frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)} \\ +\frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2}m(m-2)\sigma''(s) - m\tau'(s)]\Phi_{l,m}(s) = \lambda_l\Phi_{l,m}(s) \end{array} \tag{22}$$

we pass to a new variable $x = x(s)$ and a new function $\Psi_{l,m}(x)$ such that

$$\frac{dx}{ds} = \xi(s) \qquad\qquad \Phi_{l,m}(s) = \eta(s)\ \Psi_{l,m}(x(s)) \tag{23}$$

$\xi(s)\neq 0$ and $\eta(s)\neq 0$ for any $s\in(a,b)$, then we get the equation

$$\begin{array}{r}-\sigma(s)\,\xi^2(s)\,\ddot{\Psi}_{l,m}(x(s))-[\sigma(s)\xi'(s)+2\sigma(s)\,\xi(s)\,\frac{\eta'(s)}{\eta(s)}\\ +\tau(s)\xi(s)]\dot{\Psi}_{l,m}(x(s))+V_m(s)\Psi_{l,m}(x(s))=\lambda_l\,\Psi_{l,m}(x(s))\end{array}\tag{24}$$

where

$$\begin{array}{r}V_m(s)=\frac{m(m-2)}{4}\frac{(\sigma'(s))^2}{\sigma(s)}+\frac{m\tau(s)}{2}\frac{\sigma'(s)}{\sigma(s)}-\frac{1}{2}m(m-2)\sigma''(s)\\ -m\tau'(s)-\sigma(s)\frac{\eta''(s)}{\eta(s)}-\tau(s)\frac{\eta'(s)}{\eta(s)}\end{array}\tag{25}$$

and the dot sign means derivative with respect to x. For $\xi(s)$ and $\eta(s)$ satisfying the conditions

$$\sigma(s)\,\xi^2(s)=1\qquad\qquad \sigma(s)\xi'(s)+2\sigma(s)\,\xi(s)\,\frac{\eta'(s)}{\eta(s)}+\tau(s)\xi(s)=0.\tag{26}$$

which lead to

$$\xi(s)=\pm\frac{1}{\kappa(s)}\qquad\qquad \eta(s)=\frac{1}{\sqrt{\kappa(s)\,\varrho(s)}}\tag{27}$$

(up to a multiplicative constant), the equation (24) becomes

$$-\,\ddot{\Psi}_{l,m}(x(s))+V_m(s)\Psi_{l,m}(x(s))=\lambda_l\,\Psi_{l,m}(x(s)).\tag{28}$$

Denoting by $s(x)$ the inverse of the function $(a,b)\longrightarrow(a',b'):s\mapsto x(s)$ we get

$$\frac{ds}{dx}=\pm\kappa(s(x))\qquad\qquad \Psi_{l,m}(x)=\sqrt{\kappa(s(x))\,\varrho(s(x))}\,\Phi_{l,m}(s(x)).\tag{29}$$

The equation (28) is satisfied for any $s\in(a,b)$ if and only if

$$-\,\ddot{\Psi}_{l,m}(x)+V_m(s(x))\Psi_{l,m}(x)=\lambda_l\,\Psi_{l,m}(x)\qquad\text{for any}\quad x\in(a',b')\tag{30}$$

that is, if and only if $\Psi_{l,m}(x)$ is an eigenfunction of the Schrödinger type operator

$$\mathcal{H}_m=-\frac{d^2}{dx^2}+\mathcal{V}_m(x)\qquad\text{where}\qquad \mathcal{V}_m(x)=V(s(x)).\tag{31}$$

For each $m<\Lambda$ the functions $\Psi_{l,m}(x)$ with $m\leq l<\Lambda$ are orthogonal

$$\begin{array}{l}\int_{a'}^{b'}\overline{\Psi}_{l,m}(x)\Psi_{k,m}(x)dx=\int_a^b\overline{\Phi}_{l,m}(s(x))\Phi_{k,m}(s(x))\varrho(s(x))\,\left|\frac{ds}{dx}\right|\,dx\\ \qquad\qquad=\int_a^b\overline{\Phi}_{l,m}(s)\Phi_{k,m}(s)\varrho(s)ds=0\end{array}$$

for $k\neq l$, and satisfy the relations

$$\begin{array}{l}\mathcal{A}_m\Psi_{l,m}(x)=\left\{\begin{array}{ll}0 & \text{for } l=m\\ \Psi_{l,m+1} & \text{for } m<l<\Lambda\end{array}\right.\\ \mathcal{A}_m^+\Psi_{l,m+1}(x)=(\lambda_l-\lambda_m)\Psi_{l,m}(x)\end{array}\tag{32}$$

where

$$\mathcal{A}_m = [\kappa(s)\varrho(s)]^{1/2} A_m [\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}$$
$$\mathcal{A}_m^+ = [\kappa(s)\varrho(s)]^{1/2} A_m^+ [\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)} \tag{33}$$

are the operators corresponding to A_m and A_m^+. Particulary, we have [5]

$$\mathcal{H}_m - \lambda_m = \mathcal{A}_m^+ \mathcal{A}_m \qquad \mathcal{A}_m \mathcal{H}_m = \mathcal{H}_{m+1} \mathcal{A}_m$$
$$\mathcal{H}_{m+1} - \lambda_m = \mathcal{A}_m \mathcal{A}_m^+ \qquad \mathcal{H}_m \mathcal{A}_m^+ = \mathcal{A}_m^+ \mathcal{H}_{m+1}. \tag{34}$$

and

$$\Psi_{l,m}(x) = \frac{\mathcal{A}_m^+}{\lambda_l - \lambda_m} \frac{\mathcal{A}_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{\mathcal{A}_{l-2}^+}{\lambda_l - \lambda_{l-2}} \frac{\mathcal{A}_{l-1}^+}{\lambda_l - \lambda_{l-1}} \Psi_{l,l}(x) \tag{35}$$

for each $m \in \{0, 1, ..., l-1\}$.

Theorem 4.1. If the change of variable $s = s(x)$ is such that $ds/dx = \pm\kappa(s(x))$ then

$$\mathcal{A}_m = \pm\frac{d}{dx} + W_m(x) \qquad \mathcal{A}_m^+ = \mp\frac{d}{dx} + W_m(x) \tag{36}$$

and

$$\mathcal{V}_m(x) = W_m^2(x) \mp \dot{W}_m(x) + \lambda_m = \frac{\ddot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} + \lambda_m \tag{37}$$

where $W_m(x)$ is the superpotential [6]

$$W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} - \left(m - \frac{1}{2}\right)\frac{d\kappa}{ds}(s(x)) = \mp\frac{\dot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)}. \tag{38}$$

Proof. From $(\sigma\varrho)' = \tau\varrho$ and $ds/dx = \pm\kappa(s(x))$ we get

$$\frac{\varrho'}{\varrho} = \frac{\tau}{\kappa^2} - 2\frac{\kappa'}{\kappa} \qquad \frac{d}{ds} = \pm\frac{1}{\kappa(s(x))}\frac{d}{dx}$$

whence (36). Since $\mathcal{A}_m\Psi_{m,m} = 0$, from (31), (34) and (36) we obtain

$$\pm\dot{\Psi}_{m,m} + W_m(x)\Psi_{m,m} = 0 \qquad -\ddot{\Psi}_{m,m} + (V_m(x) - \lambda_m)\Psi_{m,m} = 0.$$

The functions

$$\psi_{l,m}(x) = \sqrt{\kappa(s(x))\,\varrho(s(x))}\,\phi_{l,m}(s(x)). \tag{39}$$

corresponding to $\phi_{l,m}$ are normalized

$$\int_{a'}^{b'} |\psi_{k,m}(x)|^2 dx = \int_a^b |\phi_{k,m}(s(x))|^2 \varrho(s(x)) \left|\frac{ds}{dx}\right| dx = \int_a^b |\phi_{k,m}(s)|^2 \varrho(s) ds = 1$$

and satisfy the relations

$$\begin{aligned} &\mathcal{A}_m\ \psi_{l,m} = \begin{cases} 0 & \text{for } l = m \\ \sqrt{\lambda_l - \lambda_m}\ \psi_{l,m+1} & \text{for } m < l < \Lambda \end{cases} \\ &\mathcal{A}_m^+\ \psi_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\ \psi_{l,m} \quad \text{for } 0 \leq m < l < \Lambda \\ &\psi_{l,m} = \frac{\mathcal{A}_m^+}{\sqrt{\lambda_l-\lambda_m}} \frac{\mathcal{A}_{m+1}^+}{\sqrt{\lambda_l-\lambda_{m+1}}} \cdots \frac{\mathcal{A}_{l-1}^+}{\sqrt{\lambda_l-\lambda_{l-1}}} \psi_{l,l}. \end{aligned} \tag{40}$$

Particular cases [1, 4, 6]. Let $\alpha_m = -(2m+\alpha-1)/2$, $\alpha'_m = (2m-\alpha-1)/2$.

(1) *Shifted oscillator*

In the case $\sigma(s) = 1$, the change of variable $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto s(x) = x$ leads to

$$\begin{aligned} W_m(x) &= -\tfrac{\alpha x+\beta}{2} \\ \mathcal{V}_m(x) &= \tfrac{(\alpha x+\beta)^2}{4} + \tfrac{\alpha}{2} + \lambda_m \end{aligned} \tag{41}$$

where $\lambda_m = -\alpha m$.

(2) *Three-dimensional oscillator*

In the case $\sigma(s) = s$, the change of variable $(0,\infty) \longrightarrow (0,\infty) : x \mapsto s(x) = x^2/4$ leads to

$$\begin{aligned} W_m(x) &= -\tfrac{\alpha}{4}x - \left(\beta + m - \tfrac{1}{2}\right)\tfrac{1}{x} \\ \mathcal{V}_m(x) &= \tfrac{\alpha^2}{16}x^2 + \left(\beta + m - \tfrac{1}{2}\right)\left(\beta + m - \tfrac{3}{2}\right)\tfrac{1}{x^2} + \tfrac{\alpha}{2}(\beta+m) + \lambda_m \end{aligned} \tag{42}$$

where $\lambda_m = -\alpha m$.

(3) *Pöschl-Teller type potential*

In the case $\sigma(s) = 1-s^2$, the change of variable $(0,\pi) \longrightarrow (-1,1) : x \mapsto s(x) = \cos x$ leads to

$$\begin{aligned} W_m(x) &= \alpha'_m \operatorname{cotan} x - \tfrac{\beta}{2} \operatorname{cosec} x = \tfrac{\alpha'_m+\beta}{2} \operatorname{cotan} \tfrac{x}{2} - \tfrac{\alpha'_m-\beta}{2} \tan\tfrac{x}{2} \\ \mathcal{V}_m(x) &= \left({\alpha'_m}^2 - \alpha'_m + \tfrac{\beta^2}{4}\right) \operatorname{cosec}^2 x - (2\alpha'_m - 1)\tfrac{\beta}{2} \operatorname{cotan} x \operatorname{cosec} x - {\alpha'_m}^2 + \lambda_m \end{aligned} \tag{43}$$

where $\lambda_m = m(m-\alpha-1)$.

(4) *Generalized Pöschl-Teller potential*

In the case $\sigma(s) = s^2-1$, the change of variable $(0,\infty) \longrightarrow (1,\infty) : x \mapsto s(x) = \cosh x$ leads to

$$\begin{aligned} W_m(x) &= \alpha_m \operatorname{cotanh} x - \tfrac{\beta}{2} \operatorname{cosech} x \\ \mathcal{V}_m(x) &= \left(\alpha_m^2 + \alpha_m + \tfrac{\beta^2}{4}\right) \operatorname{cosech}^2 x - (2\alpha_m+1)\tfrac{\beta}{2} \operatorname{cotanh} x \operatorname{cosech} x + \alpha_m^2 + \lambda_m \end{aligned} \tag{44}$$

where $\lambda_m = -m(m+\alpha-1)$.

(5) *Morse type potential*
In the case $\sigma(s)=s^2$, the change of variable $\mathbb{R}\longrightarrow(0,\infty):\ x\mapsto s(x)=\mathrm{e}^x$ leads to

$$\begin{aligned} W_m(x) &= -\tfrac{\beta}{2}\mathrm{e}^{-x}+\alpha_m \\ \mathcal{V}_m(x) &= \tfrac{\beta^2}{4}\mathrm{e}^{-2x}-(2\alpha_m+1)\tfrac{\beta}{2}\mathrm{e}^{-x}+\alpha_m^2+\lambda_m \end{aligned} \tag{45}$$

where $\lambda_m=-m(m+\alpha-1)$.

(6) *Scarf hyperbolic type potential*
In the case $\sigma(s)=s^2+1$, the change of variable $\mathbb{R}\longrightarrow\mathbb{R}:x\mapsto s(x)=\sinh x$ leads to

$$\begin{aligned} W_m(x) &= \alpha_m\tanh x-\tfrac{\beta}{2}\ \mathrm{sech}\, x \\ \mathcal{V}_m(x) &= \left(-\alpha_m^2-\alpha_m+\tfrac{\beta^2}{4}\right)\ \mathrm{sech}^2 x-(2\alpha_m+1)\tfrac{\beta}{2}\tanh x\ \mathrm{sech}\, x+\alpha_m^2+\lambda_m. \end{aligned} \tag{46}$$

where $\lambda_m=-m(m+\alpha-1)$.

5 Other shape invariant operators

In this section we restrict us [6] to the particular non-trivial cases when α and β are such that there exists $k\in\mathbb{R}$ with $\varrho(s)=\sigma^k(s)$ (see table 2).

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	k	(a,b)
s	β	$s^{\beta-1}$	$\beta-1$	$(0,\infty)$
$1-s^2$	αs	$(1-s^2)^{-\alpha/2-1}$	$-\frac{\alpha}{2}-1$	$(-1,1)$
s^2-1	αs	$(s^2-1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(1,\infty)$
s^2	αs	$s^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(0,\infty)$
s^2+1	αs	$(s^2+1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(-\infty,\infty)$

Table 2 The cases when $\varrho(s)=\sigma^k(s)$

From $(\sigma\varrho)'=\tau\varrho$ we get $\tau(s)=(k+1)\sigma'(s)=2(k+1)\kappa(s)\,\kappa'(s)$, and

$$\begin{aligned} &A_m=\kappa(s)\tfrac{d}{ds}-m\kappa'(s) \qquad A_m^+=-\kappa(s)\tfrac{d}{ds}-(2k+m+1)\kappa'(s) \\ &H_m=-\kappa^2(s)\tfrac{d}{ds^2}-2(k+1)\kappa(s)\,\kappa'(s)\tfrac{d}{ds}-m(m+2k)\kappa(s)\,\kappa''(s) \\ &\lambda_m=-m(2k+m+1)\tfrac{\sigma''(s)}{2}=-m(2k+m+1)[\kappa'^2(s)+\kappa(s)\,\kappa''(s))]. \end{aligned} \tag{47}$$

Theorem 5.1. If α and β are such that $\varrho(s)=\sigma^k(s)$ then for any $\gamma\in\mathbb{R}$ the operators

$$\tilde{A}_m=A_m+\frac{\gamma}{2m+2k+1} \qquad \tilde{A}_m^+=A_m^++\frac{\gamma}{2m+2k+1} \tag{48}$$

satisfy for $m<\Lambda-1$ with $2m+2k+1\neq 0$ the relations

$$\begin{array}{ll} \tilde{A}_m^+\tilde{A}_m=\tilde{H}_m-\tilde{\lambda}_m & \qquad \tilde{A}_m\tilde{H}_m=\tilde{H}_{m+1}\tilde{A}_m \\ \tilde{A}_m\tilde{A}_m^+=\tilde{H}_{m+1}-\tilde{\lambda}_m & \qquad \tilde{H}_m\tilde{A}_m^+=\tilde{A}_m^+\tilde{H}_{m+1} \end{array} \tag{49}$$

where

$$\tilde{H}_m=H_m-\gamma\frac{d\kappa}{ds} \qquad\qquad \tilde{\lambda}_m=\lambda_m-\frac{\gamma^2}{(2m+2k+1)^2}. \tag{50}$$

Proof. Since $A_m^+A_m=H_m-\lambda_m$ and $A_mA_m^+=H_{m+1}-\lambda_m$ we obtain

$$\begin{array}{l} (A_m^++\varepsilon)(A_m+\varepsilon)=H_m-\lambda_m-\varepsilon(2m+2k+1)\kappa'(s)+\varepsilon^2 \\ (A_m+\varepsilon)(A_m^++\varepsilon)=H_{m+1}-\lambda_m-\varepsilon(2m+2k+1)\kappa'(s)+\varepsilon^2 \end{array}$$

for any constant ε. If we choose $\varepsilon=1/(2m+2k+1)$ then we get (49)

$$\begin{array}{l} \tilde{H}_m\tilde{A}_m^+=(\tilde{A}_m^+\tilde{A}_m+\tilde{\lambda}_m)\tilde{A}_m^+=\tilde{A}_m^+(\tilde{A}_m\tilde{A}_m^++\tilde{\lambda}_m)=\tilde{A}_m^+\tilde{H}_{m+1} \\ \tilde{A}_m\tilde{H}_m=\tilde{A}_m(\tilde{A}_m^+\tilde{A}_m+\tilde{\lambda}_m)=(\tilde{A}_m\tilde{A}_m^++\tilde{\lambda}_m)\tilde{A}_m=\tilde{H}_{m+1}\tilde{A}_m. \end{array}$$

Theorem 5.2. If $0\leq m\leq l<\Lambda$ and if $\tilde{\Phi}_{l,l}$ satisfies the relation $\tilde{A}_l\tilde{\Phi}_{l,l}=0$ then

$$\tilde{\Phi}_{l,m}=\frac{\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\frac{\tilde{A}_{m+1}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{m+1}}\cdots\frac{\tilde{A}_{l-2}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{l-2}}\,\frac{\tilde{A}_{l-1}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{l-1}}\,\tilde{\Phi}_{l,l} \tag{51}$$

is an eigenfunction of $\tilde{H}_m$

$$\tilde{H}_m\tilde{\Phi}_{l,m}=\tilde{\lambda}_l\tilde{\Phi}_{l,m} \tag{52}$$

and

$$\begin{array}{l} \tilde{A}_m\tilde{\Phi}_{l,m}=\begin{cases} 0 & \text{if } m=l \\ \tilde{\Phi}_{l,m+1} & \text{if } m<l \end{cases} \\ \tilde{A}_m^+\tilde{\Phi}_{l,m+1}=(\tilde{\lambda}_l-\tilde{\lambda}_m)\tilde{\Phi}_{l,m}. \end{array} \tag{53}$$

Proof. The definition (51) of $\tilde{\Phi}_{l,m}$ can be re-written as

$$\tilde{\Phi}_{l,m}=\frac{\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} \tag{54}$$

and $\tilde{H}_l\tilde{\Phi}_{l,l}=(\tilde{A}_l^+\tilde{A}_l+\tilde{\lambda}_l)\tilde{\Phi}_{l,l}=\tilde{\lambda}_l\tilde{\Phi}_{l,l}$. The relation $\tilde{H}_m\tilde{\Phi}_{l,m}=\tilde{\lambda}_l\tilde{\Phi}_{l,m}$ follows by recurrence

$$\tilde{H}_{m+1}\tilde{\Phi}_{l,m+1}=\tilde{\lambda}_l\tilde{\Phi}_{l,m+1} \quad\Longrightarrow\quad \tilde{H}_m\tilde{\Phi}_{l,m}=\frac{\tilde{H}_m\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1}=\frac{\tilde{A}_m^+\tilde{H}_{m+1}}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1}=\tilde{\lambda}_l\tilde{\Phi}_{l,m}.$$

From the relation (54) we get

$$\tilde{A}_m\tilde{\Phi}_{l,m}=\frac{\tilde{A}_m\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1}=\frac{\tilde{H}_{m+1}-\tilde{\lambda}_m}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1}=\tilde{\Phi}_{l,m+1}.$$

If in equation (52) we pass to a new variable $x=x(s)$ such that $dx/ds=\pm 1/\kappa(s)$ and to the new functions

$$\tilde{\Psi}_{l,m}(x)=\sqrt{\kappa(s(x))\,\varrho(s(x))}\,\tilde{\Phi}_{l,m}(s(x)). \tag{55}$$

then we get the Schrödinger type equation

$$\tilde{\mathcal{H}}_m\tilde{\Psi}_{l,m}=\tilde{\lambda}_l\tilde{\Psi}_{l,m} \tag{56}$$

where

$$\tilde{\mathcal{H}}_m=-\frac{d^2}{dx^2}+\tilde{\mathcal{V}}_m(x)\qquad\text{and}\qquad\tilde{\mathcal{V}}_m(x)=\mathcal{V}_m(x)-\gamma\frac{d\kappa}{ds}(s(x)). \tag{57}$$

The operators

$$\begin{aligned}\tilde{\mathcal{A}}_m&=[\kappa(s)\varrho(s)]^{1/2}\tilde{A}_m[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}\\ \tilde{\mathcal{A}}_m^+&=[\kappa(s)\varrho(s)]^{1/2}\tilde{A}_m^+[\kappa(s)\varrho(s)]^{-1/2}|_{s=s(x)}\end{aligned} \tag{58}$$

corresponding to $\tilde{A}_m$ and $\tilde{A}_m^+$ satisfy the relations

$$\begin{aligned}&\tilde{\mathcal{A}}_m\tilde{\Psi}_{l,m}(x)=\begin{cases}0 & \text{if } m=l\\ \tilde{\Psi}_{l,m+1} & \text{if } m<l\end{cases}\\ &\tilde{\mathcal{A}}_m^+\tilde{\Psi}_{l,m+1}(x)=(\tilde{\lambda}_l-\tilde{\lambda}_m)\tilde{\Psi}_{l,m}(x)\end{aligned} \tag{59}$$

and

$$\begin{aligned}&\tilde{\mathcal{H}}_m-\tilde{\lambda}_m=\tilde{\mathcal{A}}_m^+\tilde{\mathcal{A}}_m &&\tilde{\mathcal{A}}_m\tilde{\mathcal{H}}_m=\tilde{\mathcal{H}}_{m+1}\tilde{\mathcal{A}}_m\\ &\tilde{\mathcal{H}}_{m+1}-\tilde{\lambda}_m=\tilde{\mathcal{A}}_m\tilde{\mathcal{A}}_m^+ &&\tilde{\mathcal{H}}_m\tilde{\mathcal{A}}_m^+=\tilde{\mathcal{A}}_m^+\tilde{\mathcal{H}}_{m+1}.\end{aligned} \tag{60}$$

If the change of variable $s=s(x)$ is such that $ds/dx=\pm\kappa(s(x))$ then

$$\tilde{\mathcal{A}}_m=\pm\frac{d}{dx}+\tilde{W}_m(x)\qquad\qquad\tilde{\mathcal{A}}_m^+=\mp\frac{d}{dx}+\tilde{W}_m(x) \tag{61}$$

$$\tilde{\mathcal{V}}_m(x)=\tilde{W}_m^2(x)\mp\dot{\tilde{W}}_m(x)+\tilde{\lambda}_m=\frac{\ddot{\tilde{\Psi}}_{m,m}(x)}{\tilde{\Psi}_{m,m}(x)}+\tilde{\lambda}_m \tag{62}$$

where $\tilde{W}_m(x)$ is the superpotential [6]

$$\tilde{W}_m(x)=-\frac{\tau(s(x))}{2\kappa(s(x))}-\left(m-\frac{1}{2}\right)\frac{d\kappa}{ds}(s(x))+\frac{\gamma}{2m+2k+1}=\mp\frac{\dot{\tilde{\Psi}}_{m,m}(x)}{\tilde{\Psi}_{m,m}(x)}. \tag{63}$$

Particular cases [1, 4, 6]. Let $\alpha_m = -(2m+\alpha-1)/2$, $\alpha'_m = (2m-\alpha-1)/2$.

(1) *Coulomb type potential*
In the case $\sigma(s) = s$, the change of variable $(0,\infty) \longrightarrow (0,\infty) : x \mapsto s(x) = x^2/4$ leads to

$$\begin{aligned} \tilde{W}_m(x) &= -\left(\beta+m-\tfrac{1}{2}\right)\tfrac{1}{x} + \tfrac{\gamma}{2m+2\beta-1} \\ \tilde{\mathcal{V}}_m(x) &= \left(\beta+m-\tfrac{1}{2}\right)\left(\beta+m-\tfrac{3}{2}\right)\tfrac{1}{x^2} - \gamma\tfrac{1}{x} \\ \tilde{\lambda}_m &= -\tfrac{\gamma^2}{(2m+2\beta-1)^2}. \end{aligned} \tag{64}$$

(2) *Trigonometric Rosen-Morse type potential*
In the case $\sigma(s) = 1-s^2$, the change of variable $(0,\pi) \longrightarrow (-1,1) : x \mapsto s(x) = \cos x$ leads to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha'_m \text{ cotan } x + \tfrac{\gamma}{2m-\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= \left({\alpha'_m}^2 - \alpha'_m\right) \text{ cosec}^2 x + \gamma \text{ cotan}\, x - {\alpha'_m}^2 + m(m-\alpha-1) \\ \tilde{\lambda}_m &= m(m-\alpha-1) - \tfrac{\gamma^2}{(2m-\alpha-1)^2} \end{aligned} \tag{65}$$

(3) *Eckart type potential*
In the case $\sigma(s) = s^2-1$, the change of variable $(0,\infty) \longrightarrow (1,\infty) : x \mapsto s(x) = \cosh x$ leads to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha_m \text{ cotanh } x + \tfrac{\gamma}{2m+\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= (\alpha_m^2 + \alpha_m) \text{ cosech}^2 x - \gamma \text{ cotanh}\, x + \alpha_m^2 - m(m-\alpha-1) \\ \tilde{\lambda}_m &= -m(m-\alpha-1) - \tfrac{\gamma^2}{(2m+\alpha-1)^2}. \end{aligned} \tag{66}$$

(4) *Hyperbolic Rosen-Morse type potential*
In the case $\sigma(s) = s^2+1$, the change of variable $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto s(x) = \sinh x$ leads to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha_m \tanh x + \tfrac{\gamma}{2m+\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= -(\alpha_m^2 + \alpha_m) \text{ sech}^2 x - \gamma \tanh x + \alpha_m^2 - m(m-\alpha-1) \\ \tilde{\lambda}_m &= -m(m-\alpha-1) - \tfrac{\gamma^2}{(2m+\alpha-1)^2}. \end{aligned} \tag{67}$$

6 Concluding remarks

Most of the known exactly solvable Schrödinger equations are directly related to some shape invariant operators, and most of the formulae occurring in the study of these quantum systems follow from a small number of mathematical results concerning the hypergeometric type operators. It is simpler to study these shape invariant operators then the corresponding operators occurring in various applications to quantum mechanics. Our systematic study recovers known results in a natural unified way, and allows one to extend certain results known in particular cases.

Acknowledgment

This research was supported by the grant CEx06-......

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